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A CLASS OF ETERNAL SOLUTIONS TO THE G_2 -LAPLACIAN FLOW

ANNA FINO AND ALBERTO RAFFERO

ABSTRACT. We explicitly describe the solution of the G_2 -Laplacian flow starting from an extremally Ricci-pinched closed G_2 -structure on a compact 7-manifold and we investigate its properties. In particular, we show that the solution exists for all real times and that it remains extremally Ricci-pinched. This result holds more generally on any 7-manifold whenever the intrinsic torsion of the extremally Ricci-pinched G_2 -structure has constant norm. We also discuss various examples.

1. INTRODUCTION

A G_2 -structure on a seven-dimensional smooth manifold M is characterized by the existence of a 3-form $\varphi \in \Omega^3(M)$ satisfying a suitable non-degeneracy condition. Such a 3-form gives rise to a Riemannian metric g_φ and to a volume form dV_φ on M .

By [10], the holonomy of g_φ is contained in G_2 if both $d\varphi$ and $d*_{\varphi}\varphi$ vanish, $*_{\varphi}$ being the Hodge operator of g_φ . On the other hand, when a Riemannian metric g has $\text{Hol}(g) \subseteq G_2$, then there exists a unique G_2 -structure φ satisfying $d\varphi = 0$, $d*_{\varphi}\varphi = 0$ and such that $g_\varphi = g$. A G_2 -structure defined by a non-degenerate 3-form φ which is both closed and co-closed is said to be *torsion-free* and the corresponding Riemannian metric g_φ is Ricci-flat. A G_2 -structure φ satisfying the less restrictive condition $d\varphi = 0$ is called *closed*. In such a case, the intrinsic torsion can be identified with a unique 2-form τ such that $d*_{\varphi}\varphi = \tau \wedge \varphi$, and the scalar curvature of g_φ is given by $-\frac{1}{2}|\tau|_\varphi^2$, where $|\cdot|_\varphi$ denotes the norm induced by g_φ (cf. [3]).

Closed G_2 -structures with small torsion constitute the starting point in Joyce's construction of compact 7-manifolds with holonomy G_2 [19]. Besides this and the glueing constructions [7, 21, 24], in recent years a lot of effort has been made in order to understand whether it is possible to obtain metrics with holonomy G_2 using a geometric flow approach. So far, the main results in this direction have been obtained for the G_2 -Laplacian flow introduced by Bryant in [3]:

$$\begin{cases} \frac{\partial}{\partial t}\varphi(t) = \Delta_{\varphi(t)}\varphi(t), \\ d\varphi(t) = 0, \\ \varphi(0) = \varphi. \end{cases}$$

Here, φ is a given closed G_2 -structure and $\Delta_{\varphi(t)}$ denotes the Hodge Laplacian of $g_{\varphi(t)}$.

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Short-time existence and uniqueness of the solution of the Laplacian flow on a compact manifold were proved by Bryant and Xu in [4], while the geometric and analytic properties of the flow were deeply investigated by Lotay and Wei in [30, 31, 32]. In particular, they proved that the solution $\varphi(t)$ exists as long as the velocity of the flow $|\Delta_{\varphi(t)}\varphi(t)|_{\varphi(t)}$ remains bounded. It is still an open problem whether a bound on the scalar curvature is sufficient to obtain a long-time existence result (cf. [30]). Further aspects of the Laplacian flow were studied in [9, 11, 12, 18, 25, 26, 29].

By [3], on a compact 7-manifold M the Ricci tensor and the scalar curvature of the metric induced by a closed G_2 -structure φ must satisfy the following inequality

$$\int_M [\text{Scal}(g_\varphi)]^2 dV_\varphi \leq 3 \int_M |\text{Ric}(g_\varphi)|^2 dV_\varphi.$$

In particular, the metric g_φ is Einstein if and only if the G_2 -structure is torsion-free (see also [5]). The above inequality reduces to an equality if and only if the intrinsic torsion form τ fulfills the equation $d\tau = \frac{1}{6}|\tau|_\varphi^2 \varphi + \frac{1}{6} *_\varphi (\tau \wedge \tau)$. When this happens, the closed G_2 -structure is called *extremally Ricci-pinchd* (ERP for short).

Two examples of manifolds endowed with an ERP closed G_2 -structure were obtained by Bryant [3] and Lauret [26]. Both can be described as simply connected solvable Lie groups endowed with a left-invariant ERP closed G_2 -structure. In the first case, the Lie group is not unimodular. Nevertheless, it admits a compact quotient by a torsion-free discrete subgroup of the full automorphism group of the G_2 -structure. In the second case, the Lie group is unimodular and the existence of a compact quotient has been recently proved in [23]. In [26], Lauret proved that in both examples the ERP closed G_2 -structure φ is a *steady Laplacian soliton*, i.e., it satisfies the equation

$$\Delta_\varphi \varphi = \mathcal{L}_X \varphi + \lambda \varphi,$$

for $\lambda = 0$ and for some vector field X . General results on Laplacian solitons (cf. e.g. [30, Sect. 9]) allow one to conclude that the solution of the Laplacian flow starting from one of these ERP closed G_2 -structures is self-similar and *eternal*, i.e., it exists for all real times. By [29], compact Laplacian solitons with $\lambda = 0$ are necessarily torsion-free. Thus, none of the above examples can descend to a steady Laplacian soliton on any compact quotient of the corresponding Lie group and, more generally, ERP closed G_2 -structures on compact manifolds cannot be steady Laplacian solitons.

In the present paper, we study the behaviour of the Laplacian flow starting from an ERP closed G_2 -structure in greater generality. Our main results are contained in Section 4. In Theorem 4.1, we show that the solution of the Laplacian flow starting from an ERP closed G_2 -structure φ whose intrinsic torsion form τ has constant norm is given by

$$\varphi(t) = \varphi + f(t) d\tau,$$

where $f(t) = \frac{6}{|\tau|_\varphi^2} \left(\exp \left(\frac{|\tau|_\varphi^2}{6} t \right) - 1 \right)$. From this expression, we easily see that the solution exists for all real times. This result holds, in particular, when the closed G_2 -structure is ERP and the manifold M is compact. To prove Theorem 4.1, we first show some useful results on ERP closed G_2 -structures in Proposition 4.2, and then we use them to show that the Laplacian flow we are considering is equivalent to a Cauchy problem for the function $f(t)$. The properties obtained in Proposition 4.2 allow us to conclude that the solution $\varphi(t)$

is ERP with constant velocity for all $t \in \mathbb{R}$, and that the Ricci tensor of $g_{\varphi(t)}$ is constant along the flow. Finally, by backward uniqueness and real analyticity of the solution of the Laplacian flow on compact 7-manifolds [30, 32], we conclude that a solution cannot become ERP in finite time unless it starts from an ERP closed G_2 -structure.

In Section 5, we study the asymptotic behaviour of the ERP solution $\varphi(t)$ in the compact case. In particular, we show that the volume of the manifold with respect to the Riemannian metric $g_{\varphi(t)}$ increases without bound as $t \rightarrow +\infty$, while it shrinks as $t \rightarrow -\infty$.

In Section 6, we review the two examples of ERP closed G_2 -structures mentioned above, and we discuss some related results. In Example 6.4, we show that Bryant's example belongs to a one-parameter family of inequivalent solvable Lie groups admitting a left-invariant ERP closed G_2 -structure, while in Theorem 6.8 we prove that a unimodular Lie group endowed with a left-invariant ERP closed G_2 -structure is isomorphic to Lauret's example.

2. PRELIMINARIES

2.1. Stable forms in dimension seven. According to [17], a k -form on a real n -dimensional vector space V is said to be *stable* if its $GL(V)$ -orbit is open in $\Lambda^k(V^*)$.

In the present paper, we shall mainly deal with stable 3-forms in dimension seven. They can be characterized as follows.

Proposition 2.1 ([17]). *Let V be a seven-dimensional real vector space. Consider a 3-form $\phi \in \Lambda^3(V^*)$ and the symmetric bilinear map*

$$b_\phi : V \times V \rightarrow \Lambda^7(V^*), \quad b_\phi(v, w) = \frac{1}{6} \iota_v \phi \wedge \iota_w \phi \wedge \phi.$$

Then, ϕ is stable if and only if $\det(b_\phi)^{1/9} \in \Lambda^7(V^)$ is not zero.*

Given a stable 3-form ϕ , the symmetric bilinear map

$$(2.1) \quad g_\phi := \det(b_\phi)^{-1/9} b_\phi : V \times V \rightarrow \mathbb{R}$$

is either positive definite or it has signature $(3, 4)$. These conditions characterize the only two open $GL(V)$ -orbits contained in $\Lambda^3(V^*)$.

We denote the open orbit of stable 3-forms for which (2.1) is positive definite by $\Lambda_+^3(V^*)$. It is well-known that the $GL^+(V)$ -stabilizer of a 3-form $\phi \in \Lambda_+^3(V^*)$ is isomorphic to the exceptional Lie group G_2 , and that there exists a basis (e^1, \dots, e^7) of V^* for which

$$(2.2) \quad \phi = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356},$$

e^{ijk} being a shorthand for $e^i \wedge e^j \wedge e^k$.

2.2. Closed G_2 -structures. Let M be a seven-dimensional smooth manifold and let $\Lambda_+^3(T^*M)$ denote the open subbundle of $\Lambda^3(T^*M)$ whose fibre over each point $x \in M$ is given by $\Lambda_+^3(T_x^*M)$.

A G_2 -structure on M , namely a G_2 -reduction of the frame bundle $FM \rightarrow M$, is characterized by the existence of a stable 3-form $\varphi \in \Omega_+^3(M) := \Gamma(\Lambda_+^3(T^*M))$. This 3-form gives rise to a Riemannian metric g_φ with volume form dV_φ via the identity

$$g_\varphi(X, Y) dV_\varphi = \frac{1}{6} \iota_X \varphi \wedge \iota_Y \varphi \wedge \varphi,$$

for all vector fields $X, Y \in \mathfrak{X}(M)$. We denote by $*_\varphi$ the Hodge operator determined by g_φ , and by $|\cdot|_\varphi$ the pointwise norm induced by g_φ .

It follows from the discussion in Section 2.1 that at each point x of M there exists a basis $\mathcal{B}^* = (e^1, \dots, e^7)$ of the cotangent space T_x^*M for which $\varphi|_x$ can be written as in (2.2). We shall call \mathcal{B}^* an *adapted basis* for the G_2 -structure φ .

The intrinsic torsion of a G_2 -structure φ can be identified with the covariant derivative of φ with respect to the Levi Civita connection ∇^φ of g_φ . By [10], a G_2 -structure φ is *torsion-free*, i.e., $\nabla^\varphi \varphi \equiv 0$, if and only if the 3-form φ is both closed and coclosed.

A G_2 -structure is said to be *closed* if the defining 3-form φ satisfies the equation $d\varphi = 0$. When this happens, the intrinsic torsion can be identified with a unique 2-form $\tau \in \Omega_{14}^2(M) := \{\kappa \in \Omega^2(M) \mid \kappa \wedge \varphi = -*_\varphi \kappa\} = \{\kappa \in \Omega^2(M) \mid \kappa \wedge *_\varphi \varphi = 0\}$ such that

$$d *_\varphi \varphi = \tau \wedge \varphi.$$

Clearly, the *intrinsic torsion form* τ vanishes identically if and only if the G_2 -structure is torsion-free. Notice that $\tau = d^* \varphi = -*_\varphi d *_\varphi \varphi$, thus it is coclosed and its exterior derivative coincides with the Hodge Laplacian $\Delta_\varphi \varphi = (dd^* + d^*d)\varphi = -d *_\varphi d *_\varphi \varphi$ of φ . Properties of closed G_2 -structures were investigated in [3, sect. 4.6] and in [5].

By [15], the closed 3-form φ defines a calibration on M . An oriented three-dimensional submanifold of M is called *associative* if it is calibrated by φ , while an oriented four-dimensional submanifold N is called *coassociative* if $\varphi|_N \equiv 0$ (see [15, Sect. IV] and [20, Ch. 12] for more details).

By [3], the Ricci tensor and the scalar curvature of the Riemannian metric g_φ induced by a G_2 -structure φ can be expressed in terms of the intrinsic torsion. In particular, when φ is closed the Ricci tensor has the following expression,

$$\text{Ric}(g_\varphi) = \frac{1}{4}|\tau|_\varphi^2 g_\varphi - \frac{1}{4}j_\varphi \left(d\tau - \frac{1}{2} *_\varphi (\tau \wedge \tau) \right),$$

where the map $j_\varphi : \Omega^3(M) \rightarrow \mathcal{S}^2(M)$ is defined as follows

$$j_\varphi(\beta)(X, Y) = *_\varphi (\iota_X \varphi \wedge \iota_Y \varphi \wedge \beta),$$

and the scalar curvature is given by

$$\text{Scal}(g_\varphi) = -\frac{1}{2}|\tau|_\varphi^2.$$

2.3. The G_2 -Laplacian flow. Consider a 7-manifold M endowed with a closed G_2 -structure φ . The *Laplacian flow* starting from φ is the initial value problem

$$(2.3) \quad \begin{cases} \frac{\partial}{\partial t} \varphi(t) = \Delta_{\varphi(t)} \varphi(t), \\ d\varphi(t) = 0, \\ \varphi(0) = \varphi. \end{cases}$$

This flow was introduced by Bryant in [3] to study seven-dimensional manifolds admitting closed G_2 -structures. Short-time existence and uniqueness of the solution of (2.3) when M is compact were proved in [4].

Theorem 2.2 ([4]). *Assume that M is compact. Then, the Laplacian flow (2.3) has a unique solution defined for a short time $t \in [0, \varepsilon)$, with ε depending on φ .*

Remark 2.3. The condition $d\varphi(t) = 0$ implies that the solution of (2.3) must belong to the open set

$$[\varphi]_+ := [\varphi] \cap \Omega_+^3(M)$$

in the de Rham cohomology class of φ as long as it exists.

By [30, Thm. 1.6], the solution $\varphi(t)$ exists as long as the velocity of the flow $|\Delta_{\varphi(t)}\varphi(t)|_{\varphi(t)}$ remains bounded. Moreover, if $\varphi(t)$ is defined on some interval $[0, T]$, then for each fixed time $t \in (0, T]$, $(M, \varphi(t), g_{\varphi(t)})$ is real analytic [32].

Using general results on flows of G₂-structures (see e.g. [3, 22]), it is possible to check that the evolution equation of the Riemannian metric $g_{\varphi(t)}$ induced by a G₂-structure $\varphi(t)$ evolving under the Laplacian flow is given by

$$(2.4) \quad \frac{\partial}{\partial t} g_{\varphi(t)} = -2 \operatorname{Ric}(g_{\varphi(t)}) + \frac{|\tau(t)|_{\varphi(t)}^2}{6} g_{\varphi(t)} + \frac{1}{4} \mathbf{j}_{\varphi(t)} (*_{\varphi(t)}(\tau(t) \wedge \tau(t))),$$

(see also [30]), and the corresponding volume form $dV_{\varphi(t)}$ evolves as follows

$$\frac{\partial}{\partial t} dV_{\varphi(t)} = \frac{|\tau(t)|_{\varphi(t)}^2}{3} dV_{\varphi(t)}.$$

In particular, $dV_{\varphi(t)}$ is pointwise non-decreasing.

A solution of the Laplacian flow is said to be *self-similar* if it is of the form

$$\varphi(t) = \varrho(t) F_t^* \varphi,$$

where $F_t \in \operatorname{Diff}(M)$ and $\varrho(t) \in \mathbb{R} \setminus \{0\}$ is a scaling factor. A standard argument allows one to show that the solution of the Laplacian flow is self-similar if and only if the initial datum φ satisfies the equation

$$(2.5) \quad \Delta_{\varphi} \varphi = \mathcal{L}_X \varphi + \lambda \varphi,$$

for some vector field X on M and some $\lambda \in \mathbb{R}$ (see e.g. [29, 30]). In such a case, $\varrho(t) = (1 + \frac{2}{3}\lambda t)^{3/2}$. A closed G₂-structure for which (2.5) holds is called a *Laplacian soliton*. Depending on the sign of λ , a Laplacian soliton is said to be *shrinking* ($\lambda < 0$), *steady* ($\lambda = 0$), or *expanding* ($\lambda > 0$), and the corresponding self-similar solution exists on the maximal time interval $(-\infty, -\frac{3}{2\lambda})$, $(-\infty, +\infty)$, $(-\frac{3}{2\lambda}, +\infty)$, respectively.

3. EXTREMALLY RICCI-PINCHED CLOSED G₂-STRUCTURES

Let M be a compact 7-manifold endowed with a closed G₂-structure φ . It was proved independently in [3] and [5] that the Riemannian metric g_{φ} cannot be Einstein unless φ is torsion-free. Moreover, by [3] the Ricci tensor $\operatorname{Ric}(g_{\varphi})$ and the scalar curvature $\operatorname{Scal}(g_{\varphi})$ of g_{φ} must satisfy the integral inequality

$$(3.1) \quad \int_M [\operatorname{Scal}(g_{\varphi})]^2 dV_{\varphi} \leq 3 \int_M |\operatorname{Ric}(g_{\varphi})|^2 dV_{\varphi},$$

and (3.1) reduces to an equality if and only if the intrinsic torsion form τ fulfills

$$(3.2) \quad d\tau = \frac{1}{6} |\tau|_{\varphi}^2 \varphi + \frac{1}{6} *_\varphi (\tau \wedge \tau).$$

This motivates the following.

Definition 3.1 ([3]). A closed G_2 -structure φ whose intrinsic torsion form τ satisfies (3.2) is said to be *extremally Ricci-pinchd* (ERP for short).

Useful properties of ERP closed G_2 -structures can be derived starting from (3.2) (cf. [3, Sect. 4.6]). We summarize some of them in the next proposition.

Proposition 3.2 ([3]). *Let M be a 7-manifold endowed with an ERP closed G_2 -structure φ with intrinsic torsion form $\tau \in \Omega_{14}^2(M)$ not identically vanishing. If M is compact, then τ has constant (non-zero) norm. More generally, if $|\tau|_\varphi$ is constant, then the following results hold*

- i) $\tau \wedge \tau \wedge \tau = 0$;
- ii) $\tau \wedge \tau$ is a non-zero closed simple 4-form of constant norm;
- iii) $*_\varphi(\tau \wedge \tau)$ is a non-zero closed simple 3-form of constant norm;
- iv) by points ii) and iii), the tangent bundle of M splits into the orthogonal direct sum of two integrable subbundles $TM = P \oplus Q$ with

$$P := \{X \in TM \mid \iota_X(\tau \wedge \tau) = 0\}, \quad Q := \{X \in TM \mid \iota_X *_\varphi(\tau \wedge \tau) = 0\}.$$

Moreover, the P -leaves are associative submanifolds calibrated by $-|\tau|_\varphi^{-2} *_\varphi(\tau \wedge \tau)$, while the Q -leaves are coassociative submanifolds calibrated by $-|\tau|_\varphi^{-2}(\tau \wedge \tau)$;

- v) the Ricci tensor of g_φ is given by $\text{Ric}(g_\varphi) = \frac{1}{12} \text{J}_\varphi(*_\varphi(\tau \wedge \tau)) = -\frac{1}{6} |\tau|_\varphi^2 g_\varphi|_P$. Hence, it is non-positive with eigenvalues $-\frac{1}{6} |\tau|_\varphi^2$ of multiplicity three and 0 of multiplicity four.

Further results on ERP closed G_2 -structures were obtained by Cleyton and Ivanov in [6]. We shall recall one of them in Section 6.

Remark 3.3. It follows from (3.2) and the identity $|\tau \wedge \tau|_\varphi^2 = |\tau|_\varphi^4$ (cf. [3, (2.21)]) that $d\tau$ has constant norm whenever $|\tau|_\varphi$ is constant, indeed

$$|d\tau|_\varphi^2 = \frac{1}{6} |\tau|_\varphi^4.$$

Moreover, as τ is coclosed and $d(\tau \wedge \tau) = 0$, we have

$$\Delta_\varphi \tau = - *_\varphi d *_\varphi d\tau = \frac{1}{6} |\tau|_\varphi^2 \tau.$$

4. THE LAPLACIAN FLOW STARTING FROM AN ERP CLOSED G_2 -STRUCTURE

In this section, we prove the following result.

Theorem 4.1. *Let M be a seven-dimensional manifold endowed with an ERP closed G_2 -structure φ whose intrinsic torsion form τ has constant non-zero norm. Then, the solution of the Laplacian flow starting from φ at $t = 0$ is*

$$(4.1) \quad \varphi(t) = \varphi + f(t) d\tau,$$

where

$$f(t) = \frac{6}{|\tau|_\varphi^2} \left(\exp \left(\frac{|\tau|_\varphi^2}{6} t \right) - 1 \right).$$

In particular, $\varphi(t)$ is defined for all real times, it is ERP, and the corresponding intrinsic torsion form is given by

$$\tau(t) = \exp\left(\frac{|\tau|_\varphi^2}{6}t\right)\tau.$$

Finally, the Ricci tensor of the Riemannian metric $g_{\varphi(t)}$ induced by $\varphi(t)$ is constant along the flow, i.e., $\text{Ric}(g_{\varphi(t)}) = \text{Ric}(g_\varphi)$, and $g_{\varphi(t)}$ evolves as follows

$$\frac{\partial}{\partial t}g_{\varphi(t)} = \frac{|\tau|_\varphi^2}{6}g_{\varphi(t)}\Big|_Q.$$

The proof of the first assertion in Theorem 4.1 consists in showing that the closed 3-form $\varphi(t)$ given by (4.1) defines a G_2 -structure and that the G_2 -Laplacian flow (2.3) for $\varphi(t)$ is equivalent to a Cauchy problem for the function $f(t)$. To this aim, it is useful to investigate the properties of the 3-form

$$(4.2) \quad \tilde{\varphi} := \varphi + a d\tau = \left(1 + \frac{1}{6}|\tau|_\varphi^2 a\right)\varphi + \frac{a}{6} *_\varphi(\tau \wedge \tau),$$

where φ is an ERP closed G_2 -structure with intrinsic torsion form τ of constant norm, and a is a real number. We collect them in the next result.

Proposition 4.2. *Let φ be an ERP closed G_2 -structure, assume that its intrinsic torsion form τ has constant norm, and consider the closed 3-form $\tilde{\varphi}$ given by (4.2). Then, $\tilde{\varphi}$ defines a closed G_2 -structure for all $a > -6|\tau|_\varphi^{-2}$. Whenever this happens, the following hold*

- 1) *the g_φ -orthogonal decomposition $TM = P \oplus Q$ given in point iv) of Proposition 3.2 is also $g_{\tilde{\varphi}}$ -orthogonal. Moreover, $g_{\tilde{\varphi}}|_P = g_\varphi|_P$, and $g_{\tilde{\varphi}}|_Q = (1 + \frac{1}{6}|\tau|_\varphi^2 a)g_\varphi|_Q$;*
- 2) *the volume form induced by $\tilde{\varphi}$ is $dV_{\tilde{\varphi}} = (1 + \frac{1}{6}|\tau|_\varphi^2 a)^2 dV_\varphi$;*
- 3) *the Hodge dual of $\tilde{\varphi}$ is $*_{\tilde{\varphi}}\tilde{\varphi} = (1 + \frac{1}{6}|\tau|_\varphi^2 a)(*_\varphi\varphi - \frac{a}{6}\tau \wedge \tau)$;*
- 4) *the intrinsic torsion form of $\tilde{\varphi}$ is given by $\tilde{\tau} = (1 + \frac{1}{6}|\tau|_\varphi^2 a)\tau$, it satisfies the identity $*_{\tilde{\varphi}}(\tilde{\tau} \wedge \tilde{\tau}) = *_\varphi(\tau \wedge \tau)$, and its $g_{\tilde{\varphi}}$ -norm coincides with the g_φ -norm of τ , i.e., $|\tilde{\tau}|_{\tilde{\varphi}} = |\tau|_\varphi$. Consequently, $\text{Scal}(g_{\tilde{\varphi}}) = \text{Scal}(g_\varphi)$;*
- 5) *the closed G_2 -structure $\tilde{\varphi}$ is ERP and $\text{Ric}(g_{\tilde{\varphi}}) = \text{Ric}(g_\varphi)$.*

Proof. We begin showing that $\tilde{\varphi}$ is stable for all $a \neq -6|\tau|_\varphi^{-2}$ and that it defines a G_2 -structure for all $a > -6|\tau|_\varphi^{-2}$. What we need to prove is that $\tilde{\varphi}|_x \in \Lambda_+^3(T_x M)$ for all $x \in M$. Let us fix a basis (e_1, \dots, e_7) of $T_x M$ whose dual basis (e^1, \dots, e^7) of $T_x^* M$ is an adapted basis for the closed G_2 -structure φ . Then, we have

$$\varphi|_x = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356},$$

and $g_\varphi|_x = \sum_{i=1}^7 (e^i)^2$. By point iv) of Proposition 3.2, we know that $T_x M = P_x \oplus Q_x$. Without loss of generality, we may assume that $P_x = \langle e_1, e_2, e_3 \rangle$ and $Q_x = \langle e_4, e_5, e_6, e_7 \rangle$. Since $P = \ker(\tau \wedge \tau)$, the simple 4-form $\tau \wedge \tau|_x$ must be proportional to e^{4567} . This implies that $\tau|_x \in \Lambda^2(Q_x^*)$. Consequently, as $\tau \wedge *_\varphi\varphi = 0$, there exist some real numbers c_1, c_2, c_3 for which

$$\tau|_x = c_1 (e^{45} - e^{67}) + c_2 (e^{46} + e^{57}) + c_3 (e^{47} - e^{56}),$$

In particular, $(\tau \wedge \tau)|_x = -|\tau|_\varphi^2 e^{4567}$ and $*_\varphi(\tau \wedge \tau)|_x = -|\tau|_\varphi^2 e^{123}$.

We can now compute the symmetric bilinear map $b_{\tilde{\varphi}}|_x : T_x M \times T_x M \rightarrow \Lambda^7(T_x^* M)$ and see that

$$\det(b_{\tilde{\varphi}}|_x)^{\frac{1}{9}} = \left(1 + \frac{1}{6}|\tau|_\varphi^2 a\right)^2 \det(b_\varphi|_x)^{\frac{1}{9}}.$$

Thus, the 3-form $\tilde{\varphi}|_x$ is stable if and only if $a \neq -6|\tau|_\varphi^{-2}$. Moreover, we have that

$$g_{\tilde{\varphi}}|_x = \det(b_{\tilde{\varphi}}|_x)^{-\frac{1}{9}} b_{\tilde{\varphi}}|_x = \sum_{i=1}^3 (e^i)^2 + \left(1 + \frac{1}{6}|\tau|_\varphi^2 a\right) \sum_{i=4}^7 (e^i)^2.$$

Hence, $\tilde{\varphi}|_x \in \Lambda_+^3(T_x^* M)$ if and only if $a > -6|\tau|_\varphi^{-2}$, as we claimed.

Now, assertions 1) and 2) follow immediately from the above discussion, while assertion 3) can be checked pointwise using the adapted basis for φ we are considering.

As for 4), we have

$$d *_\varphi \tilde{\varphi} = \left(1 + \frac{1}{6}|\tau|_\varphi^2 a\right) \left(d *_\varphi \varphi - \frac{a}{6} d(\tau \wedge \tau)\right) = \left(1 + \frac{1}{6}|\tau|_\varphi^2 a\right) \tau \wedge \tilde{\varphi},$$

since $\tau \wedge \tau$ is closed. Thus, by the uniqueness of the 2-form $\tilde{\tau}$ for which $d *_\varphi \tilde{\varphi} = \tilde{\tau} \wedge \tilde{\varphi}$, we obtain $\tilde{\tau} = \left(1 + \frac{1}{6}|\tau|_\varphi^2 a\right) \tau$. We can compute the Hodge dual of $\tilde{\tau}$ as follows

$$*_\varphi \tilde{\tau} = -\tilde{\tau} \wedge \tilde{\varphi} = -\left(1 + \frac{1}{6}|\tau|_\varphi^2 a\right) \tau \wedge \varphi = \left(1 + \frac{1}{6}|\tau|_\varphi^2 a\right) *_\varphi \tau.$$

Consequently, we get

$$|\tilde{\tau}|_\varphi^2 dV_{\tilde{\varphi}} = \tilde{\tau} \wedge *_\varphi \tilde{\tau} = \left(1 + \frac{1}{6}|\tau|_\varphi^2 a\right)^2 |\tau|_\varphi^2 dV_\varphi = |\tau|_\varphi^2 dV_{\tilde{\varphi}}.$$

Finally, a pointwise computation as in 3) allows us to show that $*_{\tilde{\varphi}}(\tilde{\tau} \wedge \tilde{\tau}) = *_\varphi(\tau \wedge \tau)$. Using this, it is straightforward to check that $\tilde{\varphi}$ is ERP. Now, the integrable subbundles \tilde{P} and \tilde{Q} determined by the ERP closed G_2 -structure $\tilde{\varphi}$ coincide with those determined by φ . Consequently, by point v) of Proposition 3.2 and point 1) above, we have

$$\text{Ric}(g_{\tilde{\varphi}}) = -\frac{1}{6}|\tau|_\varphi^2 g_{\tilde{\varphi}}|_{\tilde{P}} = -\frac{1}{6}|\tau|_\varphi^2 g_{\tilde{\varphi}}|_P = -\frac{1}{6}|\tau|_\varphi^2 g_\varphi|_P = \text{Ric}(g_\varphi).$$

□

We are now ready to prove the main theorem of this section.

Proof of Theorem 4.1. Consider the closed 3-form $\varphi(t) = \varphi + f(t)d\tau$, where f is a real valued smooth function such that $f(0) = 0$. Let t be small enough so that $f(t) > -6|\tau|_\varphi^{-2}$. Then, by Proposition 4.2 the 3-form $\varphi(t)$ defines a closed G_2 -structure with intrinsic torsion form $\tau(t) = \left(1 + \frac{1}{6}|\tau|_\varphi^2 f(t)\right) \tau$. Now, the Laplacian flow equation $\frac{\partial}{\partial t} \varphi(t) = \Delta_{\varphi(t)} \varphi(t)$ reads

$$\frac{d}{dt} f(t) d\tau = \left(1 + \frac{1}{6}|\tau|_\varphi^2 f(t)\right) d\tau.$$

Consequently, $\varphi(t)$ is the solution of the Laplacian flow starting from φ at $t = 0$ if and only if $f(t)$ solves the Cauchy problem

$$\begin{cases} \frac{d}{dt}f(t) = 1 + \frac{1}{6}|\tau|_\varphi^2 f(t), \\ f(0) = 0. \end{cases}$$

Thus, we have

$$f(t) = \frac{6}{|\tau|_\varphi^2} \left(\exp \left(\frac{|\tau|_\varphi^2}{6} t \right) - 1 \right),$$

whence we see that $f(t)$ is defined for all $t \in \mathbb{R}$ and that it satisfies the condition $f(t) > -6|\tau|_\varphi^{-2}$. The second part of the theorem follows immediately from points 4) and 5) of Proposition 4.2. In particular, the evolution equation of the metric $g_{\varphi(t)}$ can be obtained starting from equation (2.4) and using the identity $\frac{1}{4}j_\varphi(*_\varphi(\tau \wedge \tau)) = 3\text{Ric}(g_\varphi)$ (cf. point v) of Proposition 3.2). \square

When M is compact, backward uniqueness and real analyticity of the solution of the Laplacian flow (cf. [30, Thm. 1.4] and [32]) together with Theorem 4.1 imply the following.

Corollary 4.3. *Let $\varphi(t)$ be the solution of the Laplacian flow (2.3) on a compact 7-manifold M and assume that $\varphi(0)$ is not ERP. Then, $\varphi(t)$ cannot become ERP in finite time.*

Furthermore, from point 4) of Proposition 4.2 and Remark 3.3 we see that the velocity of the flow is constant for all $t \in \mathbb{R}$:

$$|\Delta_{\varphi(t)}\varphi(t)|_{\varphi(t)} = |d\tau(t)|_{\varphi(t)} = \frac{1}{\sqrt{6}}|\tau|_\varphi^2.$$

Since the solution $\varphi(t)$ of the Laplacian flow starting from an ERP closed G_2 -structure φ exists for all real times, and since the Ricci tensor of the corresponding metric $g_{\varphi(t)}$ does not evolve along the flow, it is natural to ask whether $\varphi(t)$ is self-similar or, equivalently, if φ is a steady Laplacian soliton. It follows from [29, Cor. 1] that the answer is negative when the manifold M is compact (see also [30, Prop. 9.5]). In detail:

Theorem 4.4 ([29]). *Let M be a compact 7-manifold. Then, the only steady Laplacian solitons on M are given by torsion-free G_2 -structures.*

On the other hand, it was recently shown in [27] that any left-invariant ERP closed G_2 -structure on a non-compact Lie group is a steady Laplacian soliton. The converse of this result does not hold, as there are examples of left-invariant steady Laplacian solitons that are not ERP, see [14].

5. ASYMPTOTIC BEHAVIOUR OF THE ERP SOLUTION

In this section, we assume that the 7-manifold M is compact and we investigate the behaviour of the ERP solution when $t \rightarrow \pm\infty$.

Using point 2) of Proposition 4.2, we obtain the following expression for the volume form induced by $\varphi(t)$

$$dV_{\varphi(t)} = \exp \left(\frac{|\tau|_\varphi^2}{3} t \right) dV_\varphi.$$

Consequently, the total volume of the compact 7-manifold M with respect to the metric $g_{\varphi(t)}$ is

$$\text{Vol}_{g_{\varphi(t)}}(M) = \int_M dV_{\varphi(t)} = \exp\left(\frac{|\tau|_{\varphi}^2}{3} t\right) \text{Vol}_{g_{\varphi}}(M),$$

whence we easily see that it increases without bound as t goes to $+\infty$, while it shrinks as t goes to $-\infty$:

$$\lim_{t \rightarrow +\infty} \text{Vol}_{g_{\varphi(t)}}(M) = +\infty, \quad \lim_{t \rightarrow -\infty} \text{Vol}_{g_{\varphi(t)}}(M) = 0.$$

We now study the behaviour of the associative P -leaves and coassociative Q -leaves along the flow. As the integrable subbundles $P(t)$ and $Q(t)$ determined by the ERP closed G_2 -structure $\varphi(t)$ coincide with the subbundles P and Q determined by $\varphi = \varphi(0)$, at each time $t \in \mathbb{R}$ we can endow the P -leaves and the Q -leaves with the Riemannian metric induced by $g_{\varphi(t)}$. We denote the corresponding Riemannian volume forms by $dV_P(t)$ and $dV_Q(t)$, respectively, and we let $dV_P := dV_P(0)$, $dV_Q := dV_Q(0)$.

Consider an oriented P -leaf $L_P \hookrightarrow M$. By point iv) of Proposition 3.2, we know that the volume form $dV_P(t)$ on L_P coincides with the restriction of the closed 3-form $-|\tau(t)|_{\varphi(t)}^{-2} *_{\varphi(t)}(\tau(t) \wedge \tau(t))$. Such a volume form is constant along the flow, as

$$dV_P = -|\tau|_{\varphi}^{-2} *_{\varphi}(\tau \wedge \tau)|_{L_P} = dV_P(t),$$

by point 4) of Proposition 4.2. Using the same result, we see that the volume form $dV_Q(t)$ of an oriented Q -leaf $L_Q \hookrightarrow M$ is given by

$$\begin{aligned} dV_Q(t) &= -|\tau(t)|_{\varphi(t)}^{-2} (\tau(t) \wedge \tau(t))|_{L_Q} = -|\tau|_{\varphi}^{-2} \exp\left(\frac{|\tau|_{\varphi}^2}{3} t\right) (\tau \wedge \tau)|_{L_Q} \\ &= \exp\left(\frac{|\tau|_{\varphi}^2}{3} t\right) dV_Q. \end{aligned}$$

It is now immediate to prove the following.

Proposition 5.1. *Let $\varphi(t)$ be the solution of the Laplacian flow starting from an ERP closed G_2 -structure on a compact 7-manifold M . Then, the volume of the P -leaves is constant along the flow. Moreover:*

1. *when $t \rightarrow +\infty$, the volume of the P -leaves goes to zero relative to the volume of the manifold, while the volume of the Q -leaves and the volume of the manifold M grow at the same rate;*
2. *when $t \rightarrow -\infty$, the volume of the Q -leaves and the volume of the manifold M tend to zero at the same rate.*

Remark 5.2. Rescaling the metric $g_{\varphi(t)}$ as $\exp\left(-\frac{|\tau|_{\varphi}^2}{6} t\right) g_{\varphi(t)}$ shows that the volume of the P -leaves goes to zero as $t \rightarrow +\infty$, that is, the P -leaves collapse as $t \rightarrow +\infty$.

6. EXAMPLES

In this section, we review two examples of 7-manifolds endowed with an ERP closed G_2 -structure obtained in [3, 26], and we discuss some related results. Further examples are considered in [2, 27, 28].

We begin with the example obtained by Bryant in [3, Ex. 1]. It consists of an invariant ERP closed G_2 -structure on the non-compact homogeneous space $SL(2, \mathbb{C}) \ltimes \mathbb{C}^2/SU(2)$. For the sake of convenience, we give the following alternative description (cf. [6, Sect. 6.3] and [26, Ex. 4.13]).

Example 6.1. Let $\mathfrak{r} = \langle e_1, e_2, e_3 \rangle$ be the three-dimensional non-unimodular solvable Lie algebra with non-zero Lie brackets

$$[e_1, e_2] = e_2, \quad [e_1, e_3] = e_3.$$

Consider the abelian Lie algebra $\mathbb{R}^4 = \langle e_4, e_5, e_6, e_7 \rangle$, and the Lie algebra homomorphism $\mu : \mathfrak{r} \rightarrow \text{Der}(\mathbb{R}^4) \cong \mathfrak{gl}(4, \mathbb{R})$ defined as follows

$$\mu(e_1) = \text{diag} \left(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right), \quad \mu(e_2) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad \mu(e_3) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

The semidirect product $\mathfrak{s} := \mathfrak{r} \ltimes_{\mu} \mathbb{R}^4$ is a seven-dimensional non-unimodular completely solvable Lie algebra. Denoted by (e^1, \dots, e^7) the dual basis of (e_1, \dots, e_7) , the structure equations $(de^i)_{i=1, \dots, 7}$ of \mathfrak{s} are given by

$$\left(0, -e^{12}, -e^{13}, \frac{1}{2}e^{14}, \frac{1}{2}e^{15}, -\frac{1}{2}e^{16} + e^{25} + e^{34}, -\frac{1}{2}e^{17} + e^{24} - e^{35} \right).$$

It is now straightforward to check that the 3-form φ given by (2.2) defines an ERP closed G_2 -structure on \mathfrak{s} with intrinsic torsion form

$$\tau = 3e^{45} - 3e^{67}.$$

From this we see that $|\tau|_{\varphi}^2 = 18$, $P = \mathfrak{r}$, and $Q = \mathbb{R}^4$. Notice that φ is exact:

$$\varphi = d \left(-\frac{1}{2}e^{23} + e^{45} - e^{67} \right).$$

Left-multiplication allows to extend the 3-form φ to a left-invariant one, say $\tilde{\varphi}$, on the simply connected solvable Lie group S with Lie algebra \mathfrak{s} . Bryant's example is then described by the pair $(S, \tilde{\varphi})$.

Remark 6.2. Albeit the Lie algebra \mathfrak{s} is not unimodular, the corresponding simply connected solvable Lie group S admits a compact quotient, as it is acted on by a torsion-free discrete subgroup $\Gamma \subset \text{Aut}(S, \tilde{\varphi})$ (cf. [3, Ex. 1] and [26, Remark 4.14]). This gives rise to a compact locally homogeneous example of ERP closed G_2 -structure on $\Gamma \backslash S$.

As observed by Cleyton and Ivanov [6], in the above example the intrinsic torsion form τ is parallel with respect to the canonical G_2 -connection $\bar{\nabla}$. More generally, the following holds.

Theorem 6.3 ([6]). *Let M be a 7-manifold endowed with a closed G_2 -structure φ and let $\overline{\nabla}$ be the corresponding canonical G_2 -connection. If the intrinsic torsion form τ of φ is parallel with respect to $\overline{\nabla}$, then φ is ERP and (M, g_φ) is locally isometric to Bryant's example.*

We now describe a one-parameter family of pairwise non-isomorphic solvable Lie algebras admitting an ERP closed G_2 -structure and including the above example.

Example 6.4. Let us consider the one-parameter family of three-dimensional non-unimodular solvable Lie algebras $\mathfrak{r}_\eta = \langle e_1, e_2, e_3 \rangle$ with non-zero Lie brackets

$$[e_1, e_2] = e_2 + \eta e_3, \quad [e_1, e_3] = -\eta e_2 + e_3, \quad \eta \in \mathbb{R}.$$

Notice that $\mathfrak{r}_0 = \mathfrak{r}$, while the number $1 + \eta^2$ provides a complete isomorphism invariant for \mathfrak{r}_η when $\eta \neq 0$ (see [33, Lemma 4.10]). Moreover, it is possible to check that the Ricci endomorphism of the inner product $g = (e^1)^2 + (e^2)^2 + (e^3)^2$ on \mathfrak{r}_η is diagonal with eigenvalue -2 of multiplicity three (cf. [33, Thm. 4.11]).

We let $\mathfrak{s}_\eta := \mathfrak{r}_\eta \ltimes_{\mu_\eta} \mathbb{R}^4$, where $\mathbb{R}^4 = \langle e_4, e_5, e_6, e_7 \rangle$ is the abelian Lie algebra, and the Lie algebra homomorphism $\mu_\eta : \mathfrak{r}_\eta \rightarrow \text{Der}(\mathbb{R}^4) \cong \mathfrak{gl}(4, \mathbb{R})$ is defined as follows

$$\mu_\eta(e_1) = \begin{pmatrix} -\frac{1}{2} & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \eta \\ 0 & 0 & -\eta & \frac{1}{2} \end{pmatrix}, \quad \mu_\eta(e_2) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad \mu_\eta(e_3) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

When $\eta \neq 0$, the seven-dimensional non-unimodular Lie algebra \mathfrak{s}_η is solvable but not completely solvable, as the complex numbers $1 \pm i\eta$, $\frac{1}{2} \pm i\eta$ are eigenvalues of ad_{e_1} . Furthermore, \mathfrak{s}_0 coincides with the Lie algebra \mathfrak{s} considered in Example 6.1.

Now, the 3-form φ given by (2.2) is exact, and it defines an ERP closed G_2 -structure on \mathfrak{s}_η with intrinsic torsion form $\tau = 3e^{45} - 3e^{67}$. By left-multiplication, we can extend φ to a left-invariant ERP closed G_2 -structure $\tilde{\varphi}_\eta$ on the simply connected solvable Lie group S_η with Lie algebra \mathfrak{s}_η . Clearly, the Lie groups S_η are pairwise non-isomorphic for all $\eta \geq 0$. However, it is possible to check that the intrinsic torsion form of the ERP closed G_2 -structure on S_η is parallel with respect to the canonical G_2 -connection. Hence, by the result of Cleyton and Ivanov recalled above, $(S_\eta, g_{\tilde{\varphi}_\eta})$ is locally isometric to Bryant's example.

Remark 6.5. Although the simply connected solvable Lie groups S_η are pairwise non-isomorphic for different values of $\eta \geq 0$, it is possible to show that the G_2 -structures $(S_\eta, \tilde{\varphi}_\eta)$ are pairwise equivalent. This has been proved in the recent work [27].

The next example was obtained by Lauret in [26]. As shown in the same paper, it is not equivalent to Bryant's example (cf. [26, Rem. 4.15]).

Example 6.6. Consider the abelian Lie algebras $\mathbb{R}^3 = \langle e_1, e_2, e_3 \rangle$ and $\mathbb{R}^4 = \langle e_4, e_5, e_6, e_7 \rangle$, and define the seven-dimensional unimodular solvable Lie algebra \mathfrak{u} as the semidirect product $\mathbb{R}^3 \ltimes_\mu \mathbb{R}^4$, with $\mu : \mathbb{R}^3 \rightarrow \text{Der}(\mathbb{R}^4) \cong \mathfrak{gl}(4, \mathbb{R})$ given by

$$\mu(e_1) = \text{diag}(1, 1, -1, -1), \quad \mu(e_2) = \text{diag}(1, -1, 1, -1), \quad \mu(e_3) = \text{diag}(1, -1, -1, 1).$$

The structure equations of \mathfrak{u} can be written with respect to the dual basis (e^1, \dots, e^7) of (e_1, \dots, e_7) as follows

$$(0, 0, 0, -e^{14} - e^{24} - e^{34}, -e^{15} + e^{25} + e^{35}, e^{16} - e^{26} + e^{36}, e^{17} + e^{27} - e^{37}).$$

The 3-form φ given by (2.2) defines an ERP closed G_2 -structure on \mathfrak{u} with intrinsic torsion form

$$\tau = -2e^{45} + 2e^{67} - 2e^{46} - 2e^{57} + 2e^{47} - 2e^{56}.$$

Notice that $|\tau|_\varphi^2 = 24$, and that P and Q coincide with \mathbb{R}^3 and \mathbb{R}^4 , respectively.

Remark 6.7. As shown in [26], in both Examples 6.1 and 6.6 the left-invariant ERP closed G_2 -structure is a steady Laplacian soliton on the corresponding non-compact simply connected solvable Lie group. By [29, Cor. 1], it cannot descend to an invariant steady Laplacian soliton on any compact quotient.

In the next theorem, we show that the Lie algebra \mathfrak{u} described in the previous example is the unique unimodular Lie algebra admitting ERP closed G_2 -structures up to isomorphism.

Theorem 6.8. *Let G be a unimodular Lie group endowed with a left-invariant ERP closed G_2 -structure. Then, its Lie algebra is isomorphic to the Lie algebra \mathfrak{u} described in Example 6.6.*

Proof. Consider the Lie algebra $\mathfrak{g} = \text{Lie}(G)$ endowed with the ERP closed G_2 -structure φ corresponding to the left-invariant one on G , and let $\mathfrak{g} = P \oplus Q$ be the induced g_φ -orthogonal decomposition of $\mathfrak{g} \cong T_{1_G}G$. Since \mathfrak{g} is unimodular and the Ricci tensor of g_φ is non-positive, the nilradical \mathfrak{n} of \mathfrak{g} is abelian by [8, Cor. 1]. Moreover, by [8, Lemma 1] we have $\text{Ric}(g_\varphi)|_{\mathfrak{n}} = 0$. Consequently, point v) of Proposition 3.2 implies that the nilradical \mathfrak{n} is contained in Q .

To prove the assertion, we consider all possible seven-dimensional unimodular Lie algebras with abelian nilradical of dimension at most four, and we show that \mathfrak{u} is the only one admitting an ERP closed G_2 -structure up to isomorphism. We shall deal with the cases \mathfrak{g} solvable and \mathfrak{g} non-solvable separately.

If \mathfrak{g} is solvable, by [36, Thm. 1]

$$\dim(\mathfrak{g}) = \dim(\mathfrak{n}) + k,$$

with $k \leq \dim(\mathfrak{n}) - \dim([\mathfrak{n}, \mathfrak{n}])$. As \mathfrak{n} is abelian, we have

$$7 = \dim(\mathfrak{n}) + k, \quad k \leq \dim(\mathfrak{n}) \leq \dim(Q) = 4,$$

whence $\mathfrak{n} = Q$. Thus, the nilradical of \mathfrak{g} is four-dimensional and abelian. Moreover, since \mathfrak{g} is solvable, the dimension of its center $\mathfrak{z}(\mathfrak{g})$ must satisfy $\dim(\mathfrak{z}(\mathfrak{g})) \leq 2\dim(\mathfrak{n}) - \dim(\mathfrak{g}) = 1$ (see e.g. [34]). We now claim that \mathfrak{g} is not decomposable. Indeed, otherwise we could write $\mathfrak{g} = \mathfrak{s}_1 \oplus \mathfrak{s}_2$ for some solvable unimodular ideals $\mathfrak{s}_1, \mathfrak{s}_2 \subseteq \mathfrak{g}$. If $\dim(\mathfrak{s}_1) = 2$, then necessarily $\mathfrak{s}_1 \cong \mathbb{R}^2$ and $\mathfrak{z}(\mathfrak{g})$ would have dimension at least two, a contradiction. If $\dim(\mathfrak{s}_1) = 3$, then \mathfrak{s}_1 is isomorphic either to the Lie algebra $\mathfrak{e}(1,1)$ of the group of rigid motions of the Minkowski 2-space or to the Lie algebra $\mathfrak{e}(2)$ of the group of rigid motions of the Euclidean 2-space (see, for instance, [1, Thm. 1.1]). Since both these Lie algebras have two-dimensional abelian nilradical, it follows that the unimodular solvable Lie algebra \mathfrak{s}_2 must be four-dimensional with two-dimensional abelian nilradical. However, there are no four-dimensional Lie algebras satisfying these properties by [1, Thm. 1.5]. Finally, if $\dim(\mathfrak{s}_1) = 1$, then $\mathfrak{s}_1 \cong \mathbb{R}$ and \mathfrak{s}_2 must be six-dimensional with three-dimensional nilradical. Any such \mathfrak{s}_2 must be decomposable by [35, Thm. 3]. We can then conclude as in the previous cases. Therefore, \mathfrak{g} is indecomposable.

Indecomposable seven-dimensional solvable Lie algebras with an abelian nilradical of dimension four were classified in [16]. A scan of all possibilities allows to conclude that only three unimodular Lie algebras of this type occur up to isomorphism. One is isomorphic to the Lie algebra \mathfrak{u} described in Example 6.6, thus it admits an ERP closed G_2 -structure. The remaining ones correspond to the fourth and the fifth pencil in [16, Prop. 5.4]. Their structure equations with respect to a suitable basis (e^1, \dots, e^7) are the following:

$$(6.1) \quad \begin{cases} de^i &= 0, \quad i = 1, 2, 3, \\ de^4 &= \alpha e^{14} - e^{15} + \gamma e^{24}, \\ de^5 &= e^{14} + \alpha e^{15} + \gamma e^{25}, \\ de^6 &= -\alpha e^{16} - \beta e^{17} - \gamma e^{26} - \rho e^{27} - \sigma e^{37}, \\ de^7 &= \beta e^{16} - \alpha e^{17} + \rho e^{26} - \gamma e^{27} + \sigma e^{36}, \end{cases}$$

where $\alpha, \beta, \rho \in \mathbb{R}$ and $\gamma, \sigma \in \mathbb{R} \setminus \{0\}$, and

$$(6.2) \quad \begin{cases} de^i &= 0, \quad i = 1, 2, 3, \\ de^4 &= \frac{1}{2}\alpha e^{14} - e^{15} + \frac{1}{2}\beta e^{24}, \\ de^5 &= e^{14} + \frac{1}{2}\alpha e^{15} + \frac{1}{2}\beta e^{25}, \\ de^6 &= -e^{36}, \\ de^7 &= -\alpha e^{17} - \beta e^{27} + e^{37}, \end{cases}$$

where $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R} \setminus \{0\}$. We now show that none of these Lie algebras admits closed G_2 -structures. The generic closed 3-form ϕ on the Lie algebra with structure equations (6.1) is given by

$$\begin{aligned} \phi &= \phi_{123}e^{123} + \phi_{124}e^{124} + \phi_{125}e^{125} + \phi_{135}e^{135} + \phi_{136}e^{136} + \phi_{137}e^{137} + \phi_{147}e^{147} + \phi_{157}e^{157} \\ &\quad + \frac{\rho}{\sigma}\phi_{357}e^{257} + \frac{\rho}{\sigma}\phi_{347}e^{247} + \frac{\alpha}{\gamma}\phi_{245}e^{145} + (\sigma\phi_{147} - \beta\phi_{347})e^{356} + (\beta\phi_{357} - \sigma\phi_{157})e^{346} \\ &\quad + (\alpha\phi_{235} - \gamma\phi_{135})e^{234} + \rho\left(\phi_{147} - \frac{\beta}{\sigma}\phi_{347}\right)e^{256} + \rho\left(\frac{\beta}{\sigma}\phi_{357} - \phi_{157}\right)e^{246} + \phi_{245}e^{245} \\ &\quad + \frac{\alpha^2\phi_{235} - \alpha\gamma\phi_{135} + \phi_{235}}{\gamma}e^{134} + \phi_{235}e^{235} + \phi_{236}e^{236} + \phi_{237}e^{237} + \phi_{347}e^{347} + \phi_{357}e^{357} \\ &\quad + \frac{\beta^2\phi_{357} - \beta\sigma\phi_{157} - \phi_{357}}{\sigma}e^{146} - \frac{\beta^2\phi_{347} - \beta\sigma\phi_{147} - \phi_{347}}{\sigma}e^{156} + \frac{\alpha}{\gamma}\phi_{267}e^{167} + \phi_{267}e^{267} \\ &\quad + \frac{\alpha\phi_{236} - \beta\phi_{237} - \gamma\phi_{136} + \rho\phi_{137}}{\sigma}e^{127} - \frac{\alpha\phi_{237} + \beta\phi_{236} - \gamma\phi_{137} - \rho\phi_{136}}{\sigma}e^{126}, \end{aligned}$$

where $\phi_{ijk} \in \mathbb{R}$. A simple computation shows that

$$b_\phi(e_6, e_6) = -\phi_{267}(\phi_{147}\phi_{357} - \phi_{157}\phi_{347})e^{1234567} = -b_\phi(e_7, e_7),$$

thus ϕ cannot define a G_2 -structure.

As for the Lie algebra with structure equations (6.2), the generic closed 3-form has the following expression

$$\begin{aligned} \phi = & \phi_{123}e^{123} + \phi_{124}e^{124} + \phi_{125}e^{125} + \phi_{135}e^{135} + \phi_{136}e^{136} + \phi_{137}e^{137} + \phi_{235}e^{235} + \phi_{236}e^{236} \\ & + \phi_{237}e^{237} + \phi_{245}e^{245} + \phi_{267}e^{267} + \phi_{346}e^{346} + \phi_{347}e^{347} + \phi_{356}e^{356} + \phi_{357}e^{357} \\ & + \frac{1}{2}(\alpha\phi_{235} - \beta\phi_{135})e^{234} + \left(\phi_{346} - \frac{\alpha}{2}\phi_{356}\right)e^{156} - \left(\phi_{347} + \frac{\alpha}{2}\phi_{357}\right)e^{157} + \alpha\phi_{267}e^{167} \\ & - \left(\phi_{356} + \frac{\alpha}{2}\phi_{346}\right)e^{146} + \left(\phi_{357} - \frac{\alpha}{2}\phi_{347}\right)e^{147} + (\alpha\phi_{237} - \beta\phi_{137})e^{127} + \alpha\phi_{245}e^{145} \\ & - \frac{\beta}{2}(\phi_{347}e^{247} + \phi_{346}e^{246} + \phi_{357}e^{257} + \phi_{356}e^{256}) + \left(\frac{\alpha^2}{2}\phi_{235} - \frac{\alpha}{2}\phi_{135} + 2\phi_{235}\right)e^{134}, \end{aligned}$$

with $\phi_{ijk} \in \mathbb{R}$, and we immediately see that also in this case $b_\phi(e_6, e_6) = -b_\phi(e_7, e_7)$.

We now focus on the case when \mathfrak{g} is unimodular and non-solvable. By the classification in [13], \mathfrak{g} is isomorphic to one of the following

$$\begin{aligned} & (-e^{23}, -2e^{12}, 2e^{13}, 0, -e^{45}, \frac{1}{2}e^{46} - e^{47}, \frac{1}{2}e^{47}); \\ & (-e^{23}, -2e^{12}, 2e^{13}, 0, -e^{45}, -\alpha e^{46}, (1 + \alpha)e^{47}), \quad -1 < \alpha \leq -\frac{1}{2}; \\ & (-e^{23}, -2e^{12}, 2e^{13}, 0, -\alpha e^{45}, \frac{\alpha}{2}e^{46} - e^{47}, e^{46} + \frac{\alpha}{2}e^{47}), \quad \alpha > 0; \\ & (-e^{23}, -2e^{12}, 2e^{13}, -e^{14} - e^{25} - e^{47}, e^{15} - e^{34} - e^{57}, 2e^{67}, 0). \end{aligned}$$

All of these Lie algebras have a three-dimensional abelian nilradical \mathfrak{n} . Since the 3-form defining an ERP closed G_2 -structure vanishes on Q , it must vanish on \mathfrak{n} . The first three Lie algebras in the above list do not admit any stable closed 3-form satisfying this condition. Indeed, in each case the abelian nilradical is given by $\mathfrak{n} = \langle e_5, e_6, e_7 \rangle$ and imposing that the generic closed 3-form ϕ satisfies $\phi(e_5, e_6, e_7) = 0$ gives $b_\phi(e_i, e_i) = 0$, for $i = 5, 6, 7$. We are then left with the last Lie algebra, whose nilradical is $\mathfrak{n} = \langle e_4, e_5, e_6 \rangle$. Let us consider the generic closed 3-form

$$\begin{aligned} \phi = & \phi_{123}e^{123} - 3\phi_{247}e^{124} - \phi_{234}e^{125} + \phi_{267}e^{126} + \phi_{127}e^{127} - \phi_{235}e^{134} + 3\phi_{357}e^{135} \\ & - \phi_{367}e^{136} + \phi_{137}e^{137} + \phi_{467}e^{146} + (\phi_{257} - \phi_{234})e^{147} - \phi_{567}e^{156} - (\phi_{235} + \phi_{347})e^{157} \\ & + 2\phi_{236}e^{167} + \phi_{234}e^{234} + \phi_{235}e^{235} + \phi_{236}e^{236} + \phi_{237}e^{237} + \phi_{247}e^{247} + \phi_{467}e^{256} \\ & + \phi_{257}e^{257} + \phi_{267}e^{267} + \phi_{567}e^{346} + \phi_{347}e^{347} + \phi_{357}e^{357} + \phi_{367}e^{367} + \phi_{456}e^{456} \\ & + \phi_{457}e^{457} + \phi_{467}e^{467} + \phi_{567}e^{567}, \end{aligned}$$

where $\phi_{ijk} \in \mathbb{R}$. The condition $\phi(e_4, e_5, e_6) = 0$ gives $\phi_{456} = 0$. Up to a basis change, we may assume that $Q = \langle e_4, e_5, e_6, v \rangle$ with $v = v_1e_1 + v_2e_2 + v_3e_3 + v_7e_7$ for some real numbers v_1, v_2, v_3, v_7 . Now, if we consider the equation $0 = \phi(e_4, e_5, v) = \phi_{457}v_7$, we see that necessarily $v_7 = 0$, otherwise $b_\phi(e_4, e_4) = 0$ and ϕ would not be stable. An inspection of the equations $\phi(e_4, e_6, v) = 0 = \phi(e_5, e_6, v)$ when ϕ is stable gives the following possibilities

- $\phi_{467} \neq 0$, $\phi_{567} \neq 0$, and $Q = \langle e_4, e_5, e_6, e_1 + \frac{\phi_{567}}{\phi_{467}}e_2 - \frac{\phi_{467}}{\phi_{567}}e_3 \rangle$;
- $\phi_{467} = 0$, $\phi_{567} \neq 0$, and $Q = \langle e_4, e_5, e_6, e_2 \rangle$;
- $\phi_{467} \neq 0$, $\phi_{567} = 0$, and $Q = \langle e_4, e_5, e_6, e_3 \rangle$.

Now, if τ is the intrinsic torsion form of an ERP closed G_2 -structure, then the 4-form $\tau \wedge \tau \in \Lambda^4(Q^*)$ must be closed and simple. However, in none of the above cases there exist closed simple 4-forms on Q . \square

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REFERENCES

- [1] A. Andrada, M. L. Barberis, I. G. Dotti, and G. P. Ovando. Product structures on four dimensional solvable Lie algebras. *Homology Homotopy Appl.* **7**(1), 9–37, 2005.
- [2] G. Ball. Seven Dimensional Geometries with Special Torsion. Ph.D. Thesis, 2019.
- [3] R. L. Bryant. Some remarks on G_2 -structures. In *Proceedings of Gökova Geometry-Topology Conference 2005*, pages 75–109. Gökova Geometry/Topology Conference (GGT), Gökova, 2006.
- [4] R. L. Bryant and F. Xu. Laplacian flow for closed G_2 -structures: Short time behavior. arXiv:1101.2004.
- [5] R. Cleyton and S. Ivanov. On the geometry of closed G_2 -structures. *Comm. Math. Phys.* **270** (1), 53–67, 2007.
- [6] R. Cleyton and S. Ivanov. Curvature decomposition of G_2 -manifolds. *J. Geom. Phys.* **58** (10), 1429–1449, 2008.
- [7] A. Corti, M. Haskins, J. Nordström, and T. Pacini. G_2 -manifolds and associative submanifolds via semi-Fano 3-folds. *Duke Math. J.* **164** (10), 1971–2092, 2015.
- [8] I. Dotti Miatello. Metrics with nonpositive Ricci curvature on semidirect products. *Q. J. Math.* **37** (147), 309–314, 1986.
- [9] M. Fernández, A. Fino, and V. Manero. Laplacian flow of closed G_2 -structures inducing nilsolitons. *J. Geom. Anal.* **26** (3), 1808–1837, 2016.
- [10] M. Fernández and A. Gray. Riemannian manifolds with structure group G_2 . *Ann. Mat. Pura Appl.* **132**, 19–45, 1982.
- [11] J. Fine and C. Yao. Hypersymplectic 4-manifolds, the G_2 -Laplacian flow and extension assuming bounded scalar curvature. *Duke Math. J.* **167**, 3533–3589, 2018.
- [12] A. Fino and A. Raffero. Closed warped G_2 -structures evolving under the Laplacian flow. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* **20** (1), 315–348, 2020.
- [13] A. Fino and A. Raffero. Closed G_2 -structures on non-solvable Lie groups. *Rev. Mat. Complut.* **32** (3), 837–851, 2019.
- [14] A. Fino and A. Raffero. Remarks on homogeneous solitons of the G_2 -Laplacian flow. To appear in *C. R. Math. Acad. Sci. Paris*.
- [15] R. Harvey and H. B. Lawson, Jr. Calibrated geometries. *Acta Math.*, **148**, 47–157, 1982.
- [16] F. Hindleleh and G. Thompson. Seven dimensional Lie algebras with a four-dimensional nilradical. *Algebras Groups Geom.* **25** (3), 243–265, 2008.
- [17] N. Hitchin. Stable forms and special metrics. In *Global differential geometry: the mathematical legacy of Alfred Gray (Bilbao, 2000)*, volume 288 of *Contemp. Math.*, pp. 70–89. Amer. Math. Soc., Providence, RI, 2001.
- [18] H. Huang, Y. Wang and C. Yao. Cohomogeneity-one G_2 -Laplacian flow on the 7-torus. *J. London Math. Soc.* **98** (2), 349–368, 2018.
- [19] D. D. Joyce. Compact Riemannian 7-manifolds with holonomy G_2 . I, II. *J. Differential Geom.* **43** (2), 291–328, 329–375, 1996.
- [20] D. D. Joyce. *Riemannian holonomy groups and calibrated geometry*, vol. 12 of *Oxford Graduate Texts in Mathematics*. Oxford University Press, Oxford, 2007.

- [21] D. D. Joyce, S. Karigiannis. A new construction of compact torsion-free G_2 -manifolds by gluing families of Eguchi-Hanson spaces. To appear in *J. Differential Geom.*
- [22] S. Karigiannis. Flows of G_2 -structures. I. *Q. J. Math.* **60** (4), 487–522, 2009.
- [23] I. Kath, J. Lauret. A new example of a compact ERP G_2 -structure. arXiv:2005.02462.
- [24] A. Kovalev. Twisted connected sums and special Riemannian holonomy. *J. Reine Angew. Math.*, **565**, 125–160, 2003.
- [25] J. Lauret. Laplacian flow of homogeneous G_2 -structures and its solitons. *Proc. Lond. Math. Soc.* **114** (3), 527–560, 2017.
- [26] J. Lauret. Laplacian solitons: questions and homogeneous examples. *Differential Geom. Appl.* **54** (B), 345–360, 2017.
- [27] J. Lauret, M. Nicolini. Extremally Ricci pinched G_2 -structures on Lie groups. arXiv:1902.06375.
- [28] J. Lauret, M. Nicolini. The classification of ERP G_2 -structures on Lie groups. To appear in *Ann. Mat. Pura Appl.*
- [29] C. Lin. Laplacian solitons and symmetry in G_2 -geometry. *J. Geom. Phys.* **64**, 111–119, 2013.
- [30] J. D. Lotay and Y. Wei. Laplacian flow for closed G_2 structures: Shi-type estimates, uniqueness and compactness. *Geom. Funct. Anal.* **27** (1), 165–233, 2017.
- [31] J. D. Lotay and Y. Wei. Stability of torsion-free G_2 structures along the Laplacian flow. *J. Differential Geom.* **111** (3), 495–526, 2019.
- [32] J. D. Lotay and Y. Wei. Laplacian flow for closed G_2 structures: real analyticity. *Comm. Anal. Geom.* **27**, 73–109, 2019.
- [33] J. Milnor. Curvatures of left invariant metrics on Lie groups. *Advances in Math.* **21** (3), 293–329, 1976.
- [34] G. M. Mubarakzjanov. On solvable Lie algebras. *Izv. Vysš. Učehn. Zaved. Matematika* **32** (1), 114–123, 1963.
- [35] J. C. Ndogmo and P. Winternitz. Solvable Lie algebras with abelian nilradicals. *J. Phys. A* **27** (2), 405–423, 1994.
- [36] L. Šnobl. On the structure of maximal solvable extensions and of Levi extensions of nilpotent Lie algebras. *J. Phys. A* **43** (50), 505202, 17, 2010.

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